An Efficient Solution Procedure for the Incompressible Navier-Stokes Equations

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This paper describes an efficient numerical method for solving the steady incompressible Navier-Stokes equations. The method is a fully implicit method based on the generalized Galerkin method, and the resulting system of equations is solved in a sweeping mode by iterative line relaxation. Results of the present method are compared with published results for separating and reattaching flows, and parameteric studies showing the effects of step size, boundary locations, and Reynolds number are presented. The present method is substantially faster than previously published methods; typical run times range from 10 sec to 1 min of 7600 CPU time; and, based on results obtained to date, it is stable at any Reynolds number.

Introduction

ALTHOUGH the boundary-layer equations are an extremely powerful tool for predicting viscous flowfields, there are certain classes of flow for which they are either inadequate or inconvenient. In the former category are flows at very low Reynolds number and flows with strongly curved streamlines at hypersonic Mach numbers, whereas the latter category includes flows with slender separation bubbles at moderate Mach numbers. This paper is directed primarily toward flows of this latter category. The primary reason for the importance of flows with separation is that "boundary-layer separation" typically constitutes an achievable upper bound on the efficiency or, indeed, utility of aerodynamic devices. The performance of wings, compressors, turbines, and jet engine inlets all are limited by boundary-layer separation.

In 1948, Goldstein[1] demonstrated that, when the freestream velocity or pressure was specified, the boundary-layer equations contain a singularity at the separation point. However, in the same paper, he remarked that "another possibility is that a singularity will always occur except for certain special pressure variations in the neighborhood of separation, and that, experimentally, whatever we may do, the pressure variations near separation will always be such that no singularity will occur." Little or no progress was made in the computation of separated flows for nearly the next 20 years. In 1964, Lees and Reeves[2] developed an integral boundary-layer method which, by virtue of computing the pressure field as a part of the solution, removed the singularity from the system of equations. However, the coupling procedure used restricted this method to supersonic flows. The basic concepts of this method were applied further in Refs. 3-5. Unfortunately, all of these methods required rather complicated iteration procedures to establish the uniqueness of the solution, and included some rather questionable assumptions with regard to the character of the flow within the separation bubble (see Ref. 6). In particular, these methods require that the maximum backflow velocity remain small.

The reason for the small backflow requirement was shown clearly by Klineberg and Steger.[7] They cast the incompressible boundary-layer equations in von Mises coordinates as

$$\frac{\partial z}{\partial x} = u\frac{\partial^2 z}{\partial \psi^2}, \quad u = \frac{\partial \psi}{\partial y}$$

where $z = u_e^2 - u^2$, x is the streamwise coordinate, and ψ is the stream function. This equation can be thought of as an unsteady heat-conduction equation with z as temperature, x as time, and u as thermal diffusivity. It is necessary that, for the integration to proceed in the positive time direction (x increasing), the diffusion coefficient be positive ($u \geq 0$). To obtain a bounded solution in regions for which $u < 0$, the direction of integration must be reversed.

This problem was solved by Klineberg and Steger,[7] Carter,[8] and Murphy et al.[9] by use of a solution procedure in which the streamwise differencing was switched in direction to agree with the local velocity direction. Implementation of this "type-dependent differencing" technique requires that the flowfield be solved by iterative sweeping or by time marching, as if it were a spatially elliptic problem rather than a parabolic one. That is to say, the entire flowfield must be computed simultaneously. References 7 and 8 restrict their treatment to incompressible laminar flows of rather artificial type, whereas Ref. 9 extends the concept to treat both incompressible and compressible flows and, in addition, the effects of turbulence. More important, perhaps, are the comparisons with both experimental data and with solutions to the Navier-Stokes equations presented in Ref. 9. In particular, these comparisons show that for flows with large streamline curvature, the effects of normal pressure gradients, ignored in classical boundary-layer theory, can modify the flow to first order, and that these effects increase with increasing Mach number.

Since these latter three solution procedures require substantial amounts of computer time because of their pseudo-elliptic nature, and since the boundary-layer equations can be shown to be physically inadequate to treat certain kinds of separations, a decision was made to develop an efficient solution procedure for the full Navier-Stokes equations. Such a procedure should yield solutions that accurately model all of the relevant physics without requiring exorbitant computational costs. The present work demonstrates the feasibility of such a procedure.

To accomplish this goal, the strategy of solution must differ markedly from that of most existing methods. The strategy used here is threefold.

1) Rather than solve the Navier-Stokes equations for the entire flowfield, as is done by most investigators, we restrict our domain of solution to that region where solution to the Navier-Stokes equations is required physically. The entire flowfield then is constructed by combining solutions of the

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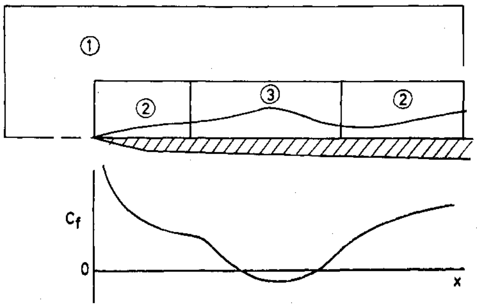


Fig. 1 Flowfield.

inviscid equations, boundary-layer equations, and Navier-Stokes equations (see Fig. 1). Region 1 is a region where the effects of viscosity are vanishingly small. Region 2 is a boundary-layer region where shear gradients in the y direction are much larger than those in the x direction and hence

$$\frac{\partial^2 u}{\partial y^2} \gg \frac{\partial^2 u}{\partial x^2}$$

Region 3 is the Navier-Stokes region where $\partial^2 u/\partial y^2$ and $\partial^2 u/\partial x^2$ may be of the same order.

2) Rather than solve the time-dependent equations with a time-accurate method, for which a lower bound on the number of time steps exists, that is, the flow establishment time divided by the largest permissable time step, we solve the steady equations which have no such physical limitations.

3) Fourth-order splined y discretization is used. This permits accurate resolution of the thin "boundary layer" with a relatively sparse nodal array, which, in turn, results in increased computational efficiency.

Analysis

Differential Equations

The incompressible, steady Navier-Stokes equations in dimensionless stream-function-vorticity variables may be written:

$$\frac{\partial \psi}{\partial y}\cdot\frac{\partial \omega}{\partial x}-\frac{\partial \psi}{\partial x}\cdot\frac{\partial \omega}{\partial y}=\frac{\partial^2 \omega}{\partial y^2}+\frac{1}{Re}\cdot\frac{\partial^2 \omega}{\partial x^2}$$

$$\omega=\frac{\partial^2 \psi}{\partial y^2}+\frac{1}{Re}\frac{\partial^2 \omega}{\partial x^2}$$

where

$$x=\frac{\bar{x}}{L} \quad u=\frac{\bar{u}}{U}=\frac{\partial \psi}{\partial y} \quad U=\bar{u}(x_0,y_e) \quad y=\frac{\bar{y}\sqrt{Re}}{L}$$

$$v=\frac{\bar{v}\sqrt{Re}}{U}=-\frac{\partial \psi}{\partial x} \quad Re=\frac{UL}{\nu} \quad L=\bar{x}_{\max}$$

and ν is the kinematic viscosity. The boundary conditions imposed on the differential equations are as follows:

At $\bar{x}=\bar{x}_0$, the flow is prescribed:

$$\psi=\psi(x_0,y)$$

$$\omega=\omega(x_0,y)$$

At $y=0$, the flow is tangent to the wall and the velocity vanishes:

$$\psi(x,0)=0$$

$$(\partial\psi/\partial y)(x,0)=0$$

At $y=y_e$, the streamwise velocity is prescribed and the vorticity is prescribed:

$$\frac{\partial \psi}{\partial y}(x,y_e)=\frac{\bar{u}_e}{U}(x,y_e)$$

$\omega(x,y_e)$ is prescribed, typically 0. At $\bar{x}=L$,

$$\frac{\partial \psi}{\partial y}\cdot\frac{\partial \omega}{\partial x}-\frac{\partial \psi}{\partial x}\cdot\frac{\partial \omega}{\partial y}=\frac{\partial^2 \omega}{\partial y^2}$$

$$\omega=\frac{\partial^2 \psi}{\partial y^2}$$

These latter conditions are that the flow satisfy the boundary-layer equations at the outflow boundary. These equations are parabolic, and hence do not permit upstream influence; in addition, they are the natural high-Reynolds-number limit of the Navier-Stokes equations.

Difference Equations

The x discretization used here is straightforward. First derivatives with respect to x are approximated by the three-point backward difference formula:

$$\left.\frac{\partial g}{\partial x}\right|_i=\frac{3g_i-4g_{i-1}+g_{i-2}}{\Delta x}+O(\Delta x^2)$$

Second derivatives with respect to x are approximated by the three-point central difference formula:

$$\left.\frac{\partial^2 g}{\partial x^2}\right|_i=\frac{g_{i+1}-2g_i+g_{i-1}}{\Delta x^2}+O(\Delta x^2)$$

In subsequent portions of this paper, the operators are written:

$$\left.\frac{\partial g}{\partial x}\right|_i=D1MMg_{i-2}+D1Mg_{i-1}+D1Cg_i$$

and

$$\left.\frac{\partial^2 g}{\partial x^2}\right|_i=D2Mg_{i-1}+D2Cg_i+D2Pg_{i+1}$$

During the development of the present program, both backward and central difference approximations for the first difference operator were considered. It was found that, if both convective terms were centrally differenced, the method was unstable. If only the term $-\partial\psi/\partial x\cdot\partial\omega/\partial y$ was centrally differenced, the method was stable. Since the results obtained appeared to be independent of the difference approximation used (except as noted), uniform three-point backward differences were used for both convective terms.

The y discretization is obtained by application of the generalized Galerkin method,[10] using splined Taylor series expansions as approximating functions and unit square waves as weighting functions (see Refs. 9 and 11). That is to say, the stream function and its derivatives are taken as:

$$\psi_{j+1}=\psi_j+\psi_j'\Delta y+\psi_j''\frac{\Delta y^2}{2!}\psi_j'''\frac{\Delta y^3}{3!}+\psi'^{v}\frac{\Delta y^4}{4!}$$

$$\psi_{j+1}'=\psi_j'+\psi_j''\Delta y+\psi_j'''\frac{\Delta y^2}{2!}+\psi_j'^{v}\frac{\Delta y^3}{3!}$$

$$\psi_{j+1}''=\psi_j''+\psi_j'''\Delta y+\psi'^{v}\frac{\Delta y^2}{2!}$$

together with the substitution

$$\psi_j^{iv} = (\psi_{j+1}''' - \psi_j''')/\Delta y$$

The vorticity and its derivatives are given by

$$\omega_{j+1} = \omega_j + \omega_j'\Delta y + \omega_j''(\Delta y^2/2!)$$

together with the substitution

$$\omega_j'' = (\omega_{j+1}' - \omega_j')/\Delta y$$

The weighting function (or basis vectors) are chosen as unit square waves so that

$$\phi_j = 1 \qquad y_j \le y \le y_{j+1}$$

$$\phi_j = 0 \qquad \text{elsewhere}$$

Substitution of the approximating functions into the differential equations and taking the inner product with respect to the weighting functions may be represented by

$$\int_j^{j+1} \psi' \frac{\partial \omega}{\partial x} dy - \int_j^{j+1} \omega' \frac{\partial \psi}{\partial x} dy - \int_j^{j+1} \omega'' + \frac{1}{Re}\frac{\partial^2 \omega}{\partial x^2} dy = 0$$

$$\int_j^{j+1} \omega dy - \int_j^{j+1} \psi'' + \frac{1}{Re}\frac{\partial^2 \psi}{\partial x^2} dy = 0$$

for $j = 1, N-1$, where N is the number of nodes in the y direction.

Substitution and formal integration, either directly or by the Taylor series expansion, yields

$$DIMM\Sigma\psi_{ij}^m XOM_{i-2} + DIM\Sigma\psi_{ij}^m XOM_{i-1} + DIC\Sigma\psi_{ij}^m XOM_i$$

$$-(DIMM\Sigma\omega_{ij}^m XPSI_{i-2} + DIM\Sigma\omega_{ij}^m XPSI_{i-1} + DIC\Sigma\omega_{ij}^m XPSI_i)$$

$$-\left(\omega_{i,j+1}' - \omega_{ij}'\right) - \frac{1}{Re}\left(D2O\Omega_{i-1}\Big|_j^{j+1} + D21\Omega_i\Big|_j^{j+1}\right.$$

$$\left.+ D22\Omega_{ij}\Big|_j^{j+1}\right) = 0$$

and

$$\Omega_i\Big|_j^{j+1} - \left(\psi_{i,j+1}' - \psi_{i,j}'\right) - \frac{1}{Re}\left(D2O\Psi_{i-1}\Big|_j^{j+1}\right.$$

$$\left.+ D21\Psi_i\Big|_j^{j+1} + D22\Psi_{i+1}\Big|_j^{j+1}\right) = 0$$

where

$$\Sigma\psi_{ij}^M XOM_i = \psi_{ij}' XOM1 + \psi_{ij}'' XOM2_i + \psi_{ij}''' XOM3_i$$

$$+ \psi_{i,j+1}''' XOM4$$

and

$$XOM1_i = \left(\omega_{ij}\Delta y + \omega_{ij}'\frac{\Delta y^2}{2} + \omega_{i,j+1}'\frac{\Delta y^2}{6}\right)$$

$$XOM2_i = \left(\omega_{ij}\frac{\Delta y^2}{2} + \omega_{ij}'\frac{5\Delta y^3}{24} + \omega_{i,j+1}'\frac{\Delta y^3}{8}\right)$$

$$XOM3_i = \left(\omega_{ij}\frac{\Delta y^3}{8} + \omega_{ij}'\frac{\Delta y^4}{18} + \omega_{i,j+1}'\frac{13\Delta y^4}{360}\right)$$

$$XOM4_i = \left(\omega_{ij}\frac{\Delta y^3}{24} + \omega_{ij}'\frac{7\Delta y^4}{360} + \omega_{i,j+1}'\frac{\Delta y^4}{72}\right)$$

and

$$\Sigma\omega_{ij}^M XPSI_i = \omega_{ij}' XPSI1_i + \omega_{ij+1}' XPSI2_i$$

$$XPSI1_i = \left(\psi_{ij}\frac{\Delta y}{2} + \psi_{ij}'\frac{\Delta y^2}{6} + \psi_{ij}''\frac{\Delta y^3}{24} + \psi_{ij}'''\frac{\Delta y^4}{144}\right.$$

$$\left.+ \psi_{ij+1}'''\frac{\Delta y^4}{720}\right)$$

$$XPSI2_i = \left(\psi_{ij}\frac{\Delta y}{2} + \psi_{ij}'\frac{\Delta y^2}{3} + \psi_{ij}''\frac{\Delta y^3}{8} + \psi_{ij}'''\frac{19\Delta y^4}{720}\right.$$

$$\left.+ \psi_{ij+1}'''\frac{\Delta y^4}{144}\right)$$

$$\Omega_i\Big|_j^{j+1} = \omega_{ij}\Delta y + \omega_{ij}'\frac{\Delta y^2}{3} + \omega_{ij+1}'\frac{\Delta y^2}{6}$$

$$\Psi i\Big|_j^{j+1} = \psi_{ij}\Delta y + \psi_{ij}'\frac{\Delta y^2}{2} + \psi_{ij}''\frac{\Delta y^3}{6} + \psi_{ij+1}'''\frac{\Delta y^4}{120}\Big)$$

The foregoing relations provide us with

$(N-1)$ Taylor series expansion in ψ
$(N-1)$ Taylor series expansions in ψ'
$(N-1)$ Taylor series expansion in ψ''
$(N-1)$ Taylor series expansions in ω
$(N-1)$ stream function equations
$(N-1)$ vorticity transport equations

or $6(N-1)$ equations in $6N$ unknowns, where N is the number of nodes in the y direction.

The four boundary conditions on the differential equation, noted earlier, provide four additional relations, leaving a deficit of two boundary conditions required by the higher-order difference approximation (see Ref. 12). These latter two conditions are taken as

$$\text{at } y = 0, \quad \omega = \psi''$$

$$\text{at } y = y_e, \quad \psi'' + \frac{1}{Re}\frac{\partial^2 \psi}{\partial x^2} = \omega(x, y_e)$$

We now have $6N$ equations in $6N$ unknowns and require a procedure for solving large systems of nonlinear equations.

In any iteration procedure, there are three elements: 1) a set of initial conditions, 2) an iteration algorithm, and 3) some criterion for terminating the iteration process. Initial conditions were generated by the program by setting $\partial^2\omega/\partial x^2 = \partial^2\psi/\partial x^2 = 0$ and marching downstream in the computational mesh. This provides a solution to the boundary-layer equations as an initial guess for the subsequent Navier-Stokes calculations. If separation is encountered during the boundary-layer pass, the last converged solution is written into the remaining field.

The iteration procedure used is a line relaxation method based on the Newton-Raphson method. The system is relaxed once at each x station, using data from the $m+1$ iteration for points to the left of x_i and data from the mth iteration for x_i and x_{i+1}. The system of equations for the iteration process takes the form $J(\phi)\Delta\phi = -\epsilon$ where $J(\phi)$ is the $6N \times 6N$ Jacobian matrix generated by differentiating each of the $6N$ equations with respect to each of the $6N$ unknowns, $\Delta\phi$ is a $6N$ column vector containing the incremental changes in each of the unknowns, and ϵ is a $6N$ column vector containing the residual error associated with each equation. After each iteration, the x index is incremented, and the procedure is repeated until the downstream boundary is reached. At this point, the entire flowfield has been updated once, the calculation then reverts to $x = x_2$ ($x = x_1$ being the initial line) and the sweeping process is repeated until the largest residual error in the system is less than some specified quantity, typically, $|\epsilon_{max}| < 10^{-2}$.

The computational process has been accelerated by noting that the Taylor series expansions for ψ, ψ', ψ'', and ω at the nodal interfaces, and the stream function equation are linear in the unknowns, and, hence, the $6N \times 6N$ Jacobian matrix may be partitioned usefully in the following manner. The iteration relation may be rewritten as

$$\begin{bmatrix} L_1 & L_2 \\ NL_1 & NL_2 \end{bmatrix} \begin{bmatrix} \Delta\phi_L \\ \Delta\phi_{NL} \end{bmatrix} = - \begin{bmatrix} \epsilon_L \\ \epsilon_{NL} \end{bmatrix}$$

where L_1 is a $5N \times 5N$ submatrix containing only coefficients that are independent of the solution and are obtained from the Taylor series expansion, the stream function equation, and boundary conditions. The matrix L_1 need be inverted only once at the beginning of the calculation. Formal manipulation and substitution yields

$$\Delta\phi_L = -L_1^{-1}(\epsilon_L + L_2\Delta\phi_{NL})$$

and

$$\Delta\phi_{NL} = (NL_2 - NL_1 L_1^{-1} L_2)^{-1}(NL_1 \cdot L_1^{-1} \cdot \epsilon_L - \epsilon_{NL})$$

The matrix that must be inverted every iteration is $(NL_2 - NL_1 \cdot L_1^{-1} \cdot L_2)$ and is only $N \times N$.

It was required initially that $|\epsilon_{max}| < 10^{-4}$, but it was found that $|\epsilon_{max}| < 10^{-2}$ would yield essentially the same solution.

Results

In this section, we describe the results of applying the computer program, embodying the foregoing analysis, to a family of retarded flows. The first set of results presented compares the results of the present method with those of Briley.[13] The second set of results presents new solutions to a family of flows similar to those of Briley, but for Reynolds numbers two orders of magnitude larger. Finally, a sequence of parametric studies is presented which demonstrates the effects of nodal spacing, boundary locations, and Reynolds number.

Comparison with Briley's Results

Briley[13] presents a sequence of solutions to the incompressible time-dependent Navier-Stokes equations. The flow configuration considered is a two-dimensional flow over a flat plate with an arbitrarily imposed streamwise velocity distribution. This velocity distribution is imposed at a constant value of $y_e = 0.0125$ and takes the form

$$u_e = 100 - 300x \qquad x < X_1$$

$$u_e = C_1 \qquad x \geq X_1$$

The values of X_1 and C_1 are adjusted to provide a family of four flows, two of which remain attached and two of which undergo separation and subsequent reattachment. In the vicinity of X_1, where the two velocity profiles are joined, a fairing is used to avoid too severe a discontinuity.

The particular advantages associated with this family of flows is that they are geometrically simple while giving rise to the interesting fluid mechanical problem of separation and reattachment. In addition, the initial portion of the flow, $x < X_1$, is the classical linearly retarded flow of Howarth,[14] which has been studied exhaustively by boundary-layer theorists (e.g., Refs. 7, 8, 14-16).

The four solutions discussed in this section correspond to Briley's solutions 1 through 4. The external velocity distributions are those described earlier; the kinematic viscosity of 0.0016, and the computational domain is taken as $x_o = 0.0167$, $x_f = 0.163$, and $y_e = 0.0125$.

Figure 2 compares the dimensionless skin friction obtained from the present method with that of Briley's solution 1. The same parameter obtained from boundary-layer theory[14] is shown for reference. The two solutions to the Navier-Stokes equations agree well everywhere, differing from each other by at most 5% of the initial value. This difference is believed to be due largely to the different manner in which the boundary conditions are imposed in the two solutions. The departure of both solutions from the boundary-layer solution is a measure of the significance of the elliptic terms $1/Re\ \partial^2\psi/\partial x^2$ and $1/Re \cdot \partial^2\omega/\partial x^2$ in the stream function and vorticity transport equations, respectively. The Navier-Stokes solutions depart from the boundary-layer solution a substantial distance upstream of the boundary-layer separation point. This early departure subsequently will be shown to result from the low Reynolds number of this flow ($Re \simeq 10^4$) rather than the separation point singularity of the boundary-layer equations (see Ref. 1).

Figure 3 compares the dimensionless skin friction obtained from the present method with that of Briley's solution 2. The x spacing used in the present method is twice that used by Briley (the effects of Δx on the solution are discussed later). The agreement between the two Navier-Stokes solutions is quite satisfactory, and their behavior relative to the boundary-layer solution is similar to that observed in Fig. 2.

Figure 4 compares the dimensionless skin friction obtained with the present method with that of Briley's solution 3. Again, agreement is quite good. The most interesting point with regard to this solution is that the imposed velocity

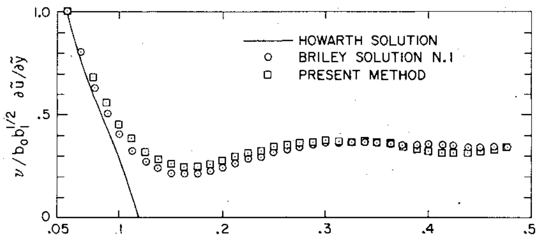


Fig. 2 Comparison of dimensionless skin friction obtained from the present method with that of Briley.

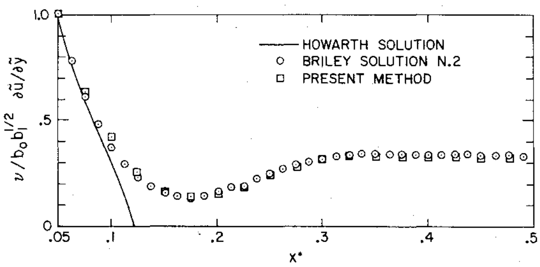


Fig. 3 Comparison of dimensionless skin friction obtained from the present method with that of Briley.

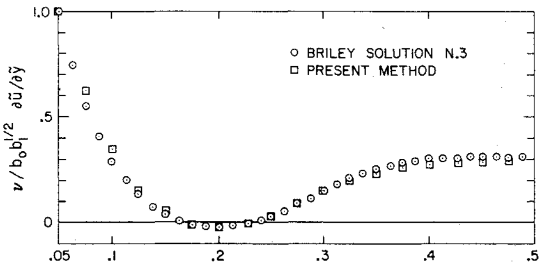


Fig. 4 Comparison of dimensionless skin friction obtained from the present method with that of Briley.

distribution is sufficiently unfavorable to cause the flow to separate. The relaxation of the velocity gradient for $x>X_l$ permits the reattachment process. As noted by Briley for his method, the present method encountered no difficulty in computing through the separation and reattachment points.

Figure 5 compares the dimensionless skin friction as obtained from the present method with that of Briley's solution 4. This flow has a large separation region and is the most severe case considered. For this case, the comparisons indicate almost a uniform 5% discrepancy, somewhat surprising in the light of the previous comparisons. It will be shown later that the outer computational boundary for this case barely encompasses the viscous portion of the flowfield. As a result of this proximity of the boundary to portions of the flow in which the solution is changing rapidly, differences in the manner in which the boundary conditions are imposed become more apparent.

High Reynolds Number Solutions for Retarded Flows

To demonstrate the capabilities of the present method at higher Reynolds numbers, a sequence of computer runs was made for flows similar to those of Briley, but with $Re \simeq 10^6$. Five separate cases were considered, each of which was a linearly retarded flow, followed by a region of constant freestream velocity. Figure 6 is a graphic display of the streamwise velocity distributions for these runs.

Figure 7 shows the distribution of dimensionless skin friction obtained for these conditions together with the same parameter as obtained by Howarth. Note that for those flows for which the adverse velocity gradient is maintained past the point of boundary-layer separation (runs $Re-3$ through $Re-5$), the Navier-Stokes solutions agree well with the boundary-layer solution very close to the separation point. It is clear, therefore, that the much earlier departure of the Navier-Stokes solutions from the boundary-layer solutions cited in connection with Figs. 2-5 is an effect attributable to the low Reynolds number of those flows.

Figure 8 compares the computed velocity profile at $x^*=0.1$ with the boundary-layer calculation of Howarth.[14] For this high Reynolds number flow, the boundary-layer solution is in substantial agreement with the Navier-Stokes solution, even in reasonably close proximity to the separation point. The overshoot in velocity in the Navier-Stokes solution because of the boundary condition ($\omega_e=0$ rather than $\partial^2\psi/\partial y_e^2=0$) is only 0.2%.

Parametric Studies

The conditions imposed on all of the runs described here are identical with those used by Briley in his solution 1, with the exceptions of the specific parameter under consideration. Figure 9 shows the sensitivity of the present method to the x step size. The two sets of points shown correspond to values of $\Delta x^*=0.0125$, the Δx^* used by Briley, and $\Delta x^*=0.0250$ (twice that used by Briley). Except for the first computed point, for which first-order streamwise differencing is used, the two solutions agree everywhere within about 2%. Briley[13] reported that, for $x^*=0.00625$, also using second-order streamwise differencing, his solutions for skin friction differed by only 0.3%. It appears then that, for these flows, any of the foregoing x step sizes provides adequate resolution of the solution.

Figure 10 shows the sensitivity of the present method to the y step size. Since unequal y spacing is used in the present method, no direct comparison with Briley's step size is

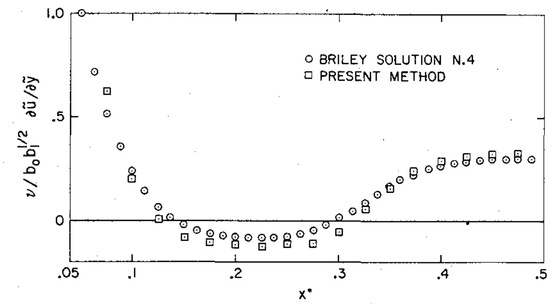


Fig. 5 Comparison of dimensionless skin friction obtained from the present method with that of Briley.

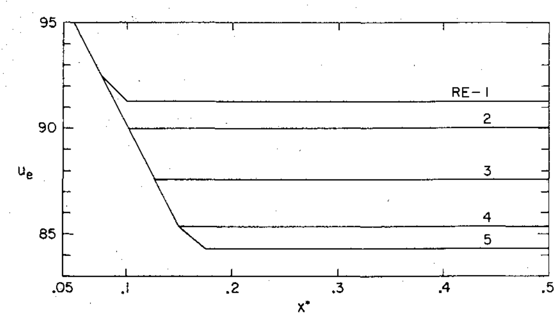


Fig. 6 Streamwise velocity distributions used in the computation of high Reynolds number retarded flows.

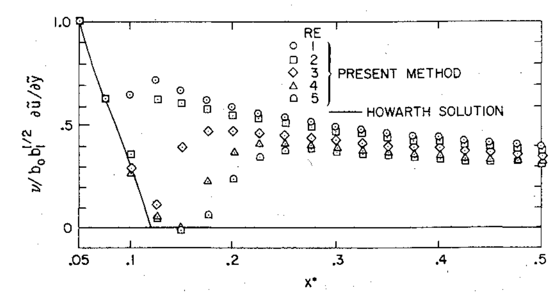
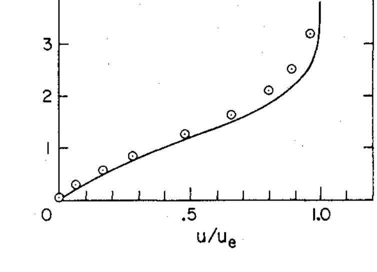


Fig. 7 Dimensionless skin friction distributions for retarded flows at high Reynolds number.

Fir. 8 Comparison velocity profile obtained from the present method with the boundary-layer solution of Howarth at $x^*=0.1$.



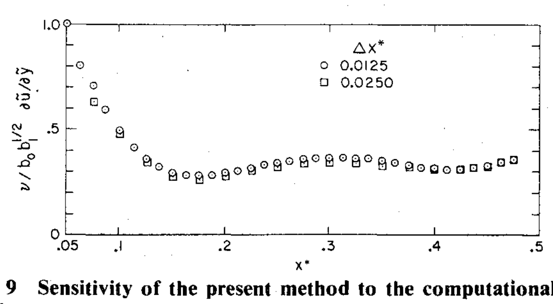


Fig. 9 Sensitivity of the present method to the computational step size in x.

possible. However, the two sets of points shown correspond in the neighborhood of the wall, to Briley's y step size and to ½ that step size. It can be seen from this figure that the differences are quite small, and hence adequate resolution is obtained. It is perhaps worth noting here that, since the present method is fourth-order accurate in the "y" direction, one requires only $\sqrt{N}$ points to obtain the same accuracy as a second-order accurate method using N points.

The next three figures show the effects of shifting the location of the outer computational boundary. The differences in the solutions shown in Fig. 11 are more dramatic in the present case than those described by Briley for two reasons. First, the flow considered here is a separated flow, so the boundary-layer thickness is much larger than that considered by Briley; second, the location of the outer boundary has been shifted by a substantially larger percentage.

Figure 12 shows a comparison of the velocity profiles at $x^* = 0.2253$ for the same family of solutions shown in Fig. 11. Again, the differences are dramatic, but in this case the tendency of the solutions to converge with increasing $\tilde{y}_e$ is more apparent. It is clear then that if $\tilde{y}_e$ is large enough, or if the solution were matched to an outer inviscid solution, there would be only one solution. This tendency to converge with increasing $\tilde{y}_e$ is equally obvious in Fig. 13, which shows the computed vorticity profiles corresponding to the velocity profiles described earlier. It is clear from these results that the lower value of $\tilde{y}_e = 0.0125$, the value used by Briley, is much too small to permit the solution to develop subject to the boundary condition $\omega = 0$ without first-order effects caused by the boundary location. It should be pointed out that this behavior in no way imples a lack of uniqueness in either the present method or that of Briley, since the location of the boundary is as much a specification of the boundary conditions as is the value of the velocity. For example, if one envisions the inviscid flow over a nonflat body, it is clear that to match that flow at differing y locations above the body, different streamwise velocity distributions must be used. Were this not the case, the unperturbed flow condition approached as $y \to \infty$ could not be satisfied.

Figure 14 compares the distributions of dimensionless skin friction for the same boundary conditions but increasing Reynolds number. The most interesting point to be made here is that for $Re = 10^5$ ($Re_x = 10^4$) the effect of Reynolds number on the converged solution is negligible, yet even at $Re = 10^6$, the elliptic terms are sufficiently strong to permit integration through the separation point without difficulty. In addition, the comparison with Howarth's solution indicates the validity of the boundary-layer equations up to some point quite near the point of separation, so long as the Reynolds number is sufficiently large.

The foregoing comparisons show that, with the exception of boundary location, variation of the parameters of the numerical method does not introduce significant changes in the solution obtained. We may conclude, then, that these solutions are in fact solutions to the differential equation.

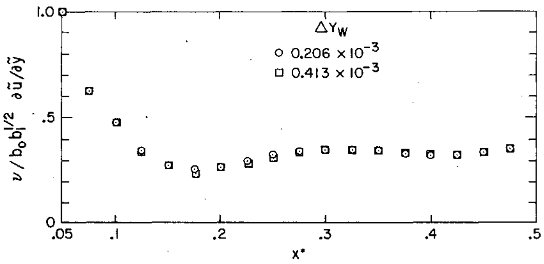


Fig. 10 Sensitivity of the present method to the computational step size in y.

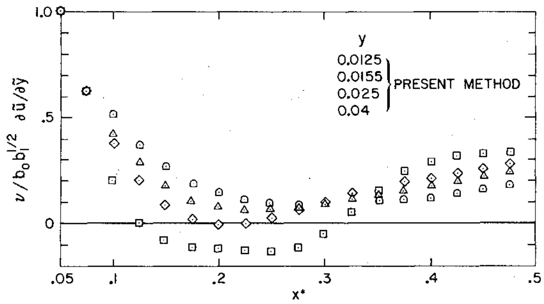


Fig. 11 Effect of boundary location on solutions of the present method.

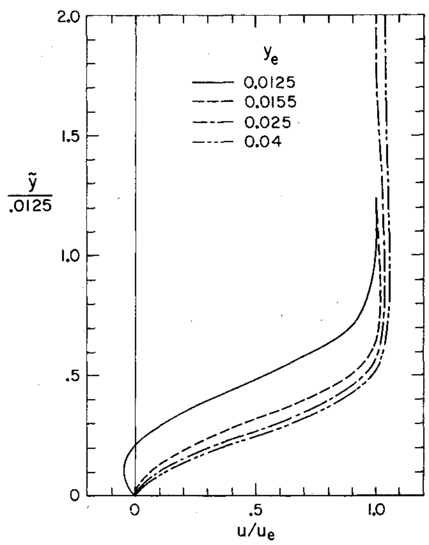


Fig. 12 Effect of displacement of outer boundary on the computed velocity profiles at $x^* = 0.2253$.

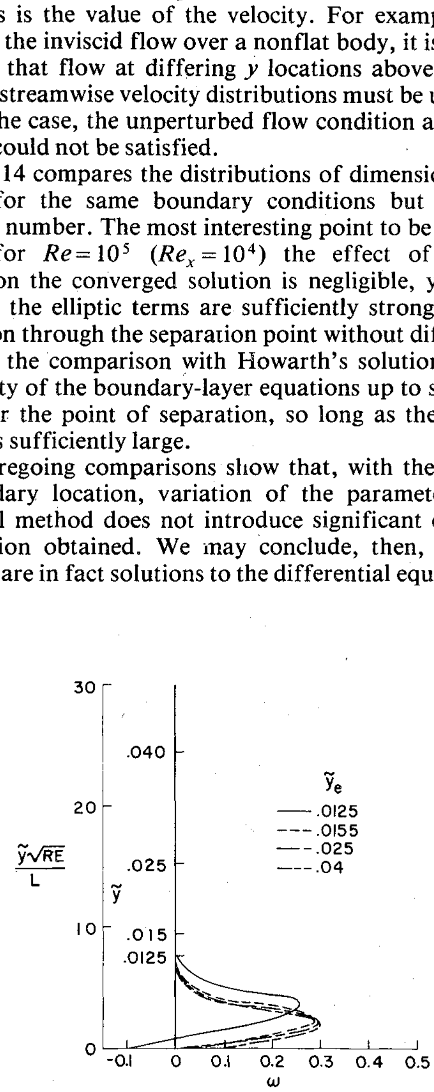


Fig. 13 Effect of displacement of outer boundary on computed vorticity profiles.

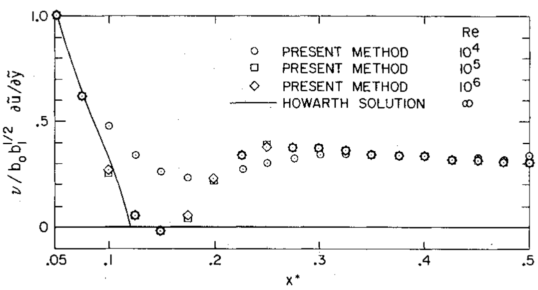


Fig. 14 Effect of Reynolds number on predicted dimensionless skin-friction distribution.

Discussion

Up to this point, we have described a new method for obtaining steady solutions to the constant-property Navier-Stokes equations, but have given no very compelling reason why the present method should be preferred over any of half a dozen other methods already available in the literature. As noted in the Introductions, the principle motivation of this study was to produce an efficient calculation method, and we have succeeded substantially in reaching this goal. Table 1 lists 19 solutions obtained with the present program, together with the CPU time and number of iterations required for convergence. Although it is difficult to make rational comparisons between the costs of various calculations schemes, it is apparent that with the present method one can obtain solutions to the full Navier-Stokes equations within a reasonable computational budget. The only comparative information we have is that Briley's solution 4 required 45 min of Univac 1108 time compared to 13 sec for the present method on a CDC 7600. A more recent version of Brileys method[17] indicates a substantial improvement in computational efficiency associated with the modification of the treatment of wall boundary conditions. However, since this new version was not applied to any member of the same family of flows considered earlier, it is impossible to quantify the improvement.

The reasons underlying the solution efficiency stem from the elements of the strategy of solution.

1) The vorticity transport and stream function equations are solved as a coupled system so that no double iteration loop is required.

2) The y discretization is of higher order so that fewer mesh points may be used for the same accuracy.

3) The steady-state equations are solved so that, for example, in case 10 corresponding to Briley's solution 4, convergence was obtained in 43 iterations rather than 900 time steps required by a typical time-accurate approach to the steady state.

4) The initial conditions are generated from the boundary-layer equations which are the infinite-Reynolds-number limit of the equations solved.

A few remarks on the applicability of the present method to more general classes of flows may be appropriate here. It is clear that the present method is tailored to treat slender separation bubbles of the boundary-layer type, and to the extent that the method is written in Cartesian coordinates with the boundary conditions specified on, at least piecewise, constant values of y, it is restricted to such flows. The difference schemes and iteration procedures clearly can be applied to arbitrary coordinate systems, but such changes would require the development of a new computer code rather than extension of the present one. Briefly, the method provides a *direct* solution procedure for the classes of flows treated in Refs. 7 and 8 and can treat regions of substantially larger reversed flow than either of the above cited methods.

Concluding Remarks

Although the foregoing presentation is not exhaustive, it does provide a reasonable insight into the background and capabilities of the present method. As it now stands, the method is capable of rigorously treating only incompressible, laminar flows over flat plate configurations, although with arbitrary pressure gradients. Extension of the method to the treatment of turbulent flows requires, in principle, only the introduction of a model for the turbulent shear stress (e.g., and eddy viscosity), although, in general, the utility of such a relation must remain somewhat questionable while the field turbulent modeling is in its current state of flux. Equilibrium flows can be treated adequately using existing simple models,

The generation of a method similar to the present method to treat compressible flows is under active investigation and appears to be conceptually straightforward so long as the flow is shock-free. The use of continuous functions to represent the solution would require that a shock-fitting procedure be used to account for discontinuities in the solution. Although this clearly could be done, it undoubtedly would be at some cost to the efficiency of the method.

Among the conclusions to be reached with regard to the present method are the following.

1) Generally, because of the solution strategy used, the goal of developing an efficient computational procedure for the Navier-Stokes equations has been attained.

2) This efficiency cannot be attributed to any single element of the strategy, but results from the particular combination of elements. The restricted computational domain, together with the boundary conditions, permit solution of the Navier-Stokes equations only in regions for which the boundary-layer and inviscid equations are invalid. The transformation and discretization procedures used provide stable calculations of high accuracy at arbitrary Reynolds numbers. The solution of the steady, rather than time-dependent, coupled equations provides rapid convergence.

3) The accuracy of the method is comparable to that of existing methods, and because of the efficiency of the method,

Table 1 Execution time

Run No.	Nodes $M \times N$	Comments	y_e	Iterations to converge	CDC 7600 CPU time, sec
1	19×18	Briley No. 1	0.0155	15	12.83
2	35×18	Briley No. 1	0.0155	33	45.64
4	19×13	Briley No. 1	0.0155	15	5.82
5	19×13	Briley No. 1—but $Re=10^5$	0.0049	12	4.93
6	35×17	Briley No. 1	0.0125	5	3.57
$Re-1$	19×13	$Re=10^6$, $u_e=91.21$, $x=X_I$	0.0025	5	3.57
$Re-2$	19×13	$Re=10^6$, $u_e=89.99$, $x=X_I$	0.0025	9	4.04
$Re-3$	19×15	$Re=10^5$, $u_e=87.49$, $x=X_I$	0.0035	9	5.56
$Re-4$	19×19	$Re=10^6$, $u_e=85.30$, $x=X_I$	0.0065	11	13.91
$Re-5$	19×20	$Re=10^6$, $u_e=84.25$, $x=X_I$	0.0085	11	15.91
7	18×12	Birley No. 2	0.0125	14	3.64
8	18×13	Birley No. 2	0.0155	15	5.57
9	18×12	Briley No. 3	0.0125	16	4.89
10	18×12	Briley No. 4	0.0125	43	13.14
11	18×13	Briley No. 4	0.0155	17	6.13
12	18×15	Briley No. 4	0.025	27	15.40
13	18×18	Briley No. 4	0.04	48	33.34
14	18×12	Briley No. 4—like 10 but with type dependent streamwise differencing	0.0125	42	12.79

the effects of mesh size, difference formulation, and boundary locations can be investigated parametrically within a reasonable computational budget.

4) The results and comparisons presented indicate that the present method is stable at high Reynolds numbers and, taken together with the results of Ladyzhenskaya,[18] that the two-dimensional time dependent Navier-Stokes equations with smooth boundary conditions have a unique solution, show that instabilities encountered by earlier investigators were of numerical rather than physical origin. In the authors experience with the method, convergence to plausible solutions always is obtained when reasonable choices are made for boundary conditions and locations. In the absence of standard solutions for comparison, however, the accuracy of solutions for more exotic boundary conditions cannot be assessed.

5) Extension of the present method to treat more general flow configurations, i.e., shock-free compressible and turbulent flows, appears to present no major conceptual difficulties.

6) More generally, comparison of the results of the present method with those of boundary-layer theory, at high Reynolds numbers, indicates that classical boundary-layer theory is valid in regions quite near separation.

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